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Q.1] For a parabolic cylindrical coordinates (u, v, z) the Cartesian coordinate transformations are given as $x = \frac{1}{2}(u^2 - v^2)$, $y = uv$, $z = z$ where $-\infty < u < \infty$, $v \geq 0$ and $-\infty < z < \infty$. Obtain the scale factors h_u, h_v & h_z.

Soln. **I^{st} Method:** Position vector $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

or $\vec{R} = \frac{1}{2}(u^2 - v^2)\hat{i} + uv\hat{j} + z\hat{k}$

Now $d\vec{R} = \frac{\partial \vec{R}}{\partial u} du + \frac{\partial \vec{R}}{\partial v} dv + \frac{\partial \vec{R}}{\partial z} dz$

or $d\vec{R} = (u\hat{i} + v\hat{j})du + (-v\hat{i} + u\hat{j})dv + \hat{k}\,dz$

$= (u\,du - v\,dv)\hat{i} + (v\,du + u\,dv)\hat{j} + \hat{k}\,dz$

$ds^2 = d\vec{R}\cdot d\vec{R} = \left[(u\,du - v\,dv)\hat{i} + (v\,du - u\,dv)\hat{j} + \hat{k}dz\right]\cdot\left[(u\,dv - v\,dv)\hat{i} + (v\,du - u\,dv)\hat{j} + \hat{k}dz\right]$

$= u^2(du)^2 + v^2(dv)^2 + v^2(du)^2 + u^2(dv)^2 + (dz)^2$

$ds^2 = (u^2 + v^2)(du)^2 + (u^2 + v^2)dv^2 + (dz)^2$

$ds^2 = h_u^2(du)^2 + h_v^2(dv)^2 + h_z^2(dz)^2$

From here we obtain $\boxed{h_u = \sqrt{u^2 + v^2},\ h_v = \sqrt{u^2 + v^2},\ h_z = 1}$

2^{nd} Method: By calculating the square of the element of arc length $ds^2 = dx^2 + dy^2 + dz^2$

Since $x = \frac{1}{2}(u^2 - v^2)$, $y = uv$ and $z = z$.

$\Rightarrow$ $dx = u\,du - v\,dv$, $dy = u\,dv + v\,du$, $dz = dz$

Now $ds^2 = dx^2 + dy^2 + dz^2$

$= (u\,du - v\,dv)^2 + (u\,dv + v\,du)^2 + dz^2$

$= (u^2+v^2)(du)^2 + (u^2+v^2)(dv)^2 + (dz)^2$

$ds^2 = h_u^2\,du^2 + h_v^2\,dv^2 + h_z^2\,dz^2$

or $\boxed{h_u = h_v = \sqrt{u^2+v^2} \text{ \& } h_z = 1}$

[Q.2] For oblate spheroidal coordinate (ξ, η, ϕ) the cartesian coordinate transformation rules are given as

$x = a\cosh\xi\cos\eta\cos\phi$, $y = a\cosh\xi\cos\eta\sin\phi$,

$z = a\sinh\xi\sin\eta$. Obtain the scale factors h_ξ, h_η and h_ϕ. Note that $\xi \geq 0$, $-\frac{\pi}{2} \leq \eta \leq \frac{\pi}{2}$, $0 \leq \phi < 2\pi$.

Soln. From the transformation rule we calculate dx, dy & dz.

$dx = -a\cosh\xi\cos\eta\sin\phi\,d\phi - a\cosh\xi\sin\eta\cos\phi\,d\eta$
$\quad + a\sinh\xi\cos\eta\cos\phi\,d\xi$

$dy = a\cosh\xi\cos\eta\cos\phi\,d\phi - a\cosh\xi\sin\eta\sin\phi\,d\eta$
$\quad + a\sinh\xi\cos\eta\sin\phi\,d\xi$

$dz = a\sinh\xi\cos\eta\,d\eta + a\cosh\xi\sin\eta\,d\xi$

Now $ds^2 = dx^2 + dy^2 + dz^2$

$$= a^2\Big[\big(\cosh\xi\cos\eta\sin\phi\,d\phi - \cosh\xi\sin\eta\sin\phi\,d\eta + \sinh\xi\cos\eta\cos\phi\,d\xi\big)^2 + \big(\cosh\xi\cos\eta\cos\phi\,d\phi - \cosh\xi\sin\eta\sin\phi\,d\eta + \sinh\xi\cos\eta\sin\phi\,d\xi\big)^2 + \big(\sinh\xi\cos\eta\,d\eta + \cosh\xi\sin\eta\,d\xi\big)^2\Big]$$

use $(a+b+c)^2 = a^2+b^2+c^2 + 2(ab+bc+ca)$

H.W. Algebra & hyperbolic function identities

$= a^2(\sinh^2\xi + \sin^2\eta)\,d\xi^2$
$\quad + a^2(\sinh^2\xi + \sin^2\eta)\,d\eta^2$
$\quad + a^2\cosh^2\xi\cos^2\eta\,d\phi^2$

or $\boxed{h_\xi = h_\eta = a\sqrt{\sinh^2\xi + \sin^2\eta},\ h_\phi = a\cosh\xi\cos\eta}$